

Study Notes on Lie Theory and Its Applications in Robotics

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March 5, 2026

Disclaimer and Acknowledgments

Accuracy Warning

Use **with caution**. These notes are a condensed synthesis intended for study and quick reference. They may contain:

- Mathematical errors or typos.
- Simplifications that sacrifice rigorous completeness for intuition.
- Notational inconsistencies.

They are **not** a substitute for peer-reviewed literature or official documentation. Verify all formulas before implementation in production control loops.

Primary Attribution

The theoretical framework, mathematical derivations, some figures, and notational style presented in this document are fundamentally derived from the seminal work of Joan Solà et al.

Source: Solà, Joan, Jeremie Deray, and Dinesh Atchuthan. "A micro Lie theory for state estimation in robotics." *arXiv preprint arXiv:1812.01537* (2018). [4]

Readers are strongly encouraged to consult the original manuscript [4] for authoritative proofs and extended discourse. Any deviation from Solà's rigor is an error on the part of the author of these notes.

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1 Introduction

Welcome to these study notes on Lie theory and its pivotal role in modern robotics. This document aims to provide a comprehensive yet accessible overview of Lie groups, Lie algebras, and their practical applications, particularly in state estimation. We will delve into the mathematical foundations, exploring why traditional vector space approaches fall short for representing quantities like rotations and poses, and how Lie theory offers a rigorous and elegant solution.

1.1 Why Lie Theory in Robotics?

Robotics inherently deals with continuous motion and transformations. Representing and manipulating these quantities, such as orientations (rotations) and positions (poses), requires mathematical tools that go beyond standard linear algebra. Lie theory provides the necessary framework to handle the non-Euclidean geometry of these transformations, enabling robust and accurate algorithms for navigation, control, and state estimation.

2 Mathematical Background

To effectively model and solve problems in robotics, especially those involving state estimation, a solid understanding of fundamental mathematical spaces and transformations is essential. This section introduces key concepts such as vector spaces, Euclidean spaces, and manifolds, and discusses the implications of their properties for robotic applications.

2.1 Vector Space

A *vector space* (or linear space) is a fundamental algebraic structure comprising a set of objects, called vectors, that can be added together and multiplied by scalars. Formally, a vector space V over a *field* F is a set V equipped with two binary operations:

- (a) **Vector Addition:** A function $+$: $V \times V \rightarrow V$, which combines two vectors $\mathbf{u}, \mathbf{v} \in V$ to produce another vector $\mathbf{u} + \mathbf{v} \in V$.
- (b) **Scalar Multiplication:** A function \cdot : $F \times V \rightarrow V$, which combines a scalar $\alpha \in F$ and a vector $\mathbf{v} \in V$ to produce another vector $\alpha \cdot \mathbf{v} \in V$.

These operations must satisfy a set of axioms, including closure (results of operations stay within V), associativity, commutativity for vector addition, existence of a unique additive identity (the zero vector $\mathbf{0}$), existence of a unique additive inverse for every vector, distributivity properties, and compatibility with scalar multiplication. A key characteristic of vector spaces is their inherent *linearity*, meaning that combinations of vectors behave predictably under these operations.

2.2 Euclidean Space

A *Euclidean space* is a specific type of vector space that is endowed with additional structure, allowing for the concepts of length, angle, and distance. Specifically, an n -dimensional Euclidean space, often denoted \mathbb{R}^n , is a vector space over the field of real numbers \mathbb{R} equipped with a *positive-definite inner product* $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$. The inner product for vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is commonly the dot product, $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v}$.

The inner product enables the definition of:

- **Norm (or Length):** The length of a vector \mathbf{v} is $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.
- **Distance:** The distance between two vectors (or points) \mathbf{u} and \mathbf{v} is $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.
- **Angle:** The angle θ between two non-zero vectors \mathbf{u} and \mathbf{v} is given by $\cos \theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|}$.

Euclidean spaces are characterized by their "flatness," which implies that the shortest path between any two points is a straight line, and the laws of Euclidean geometry (e.g., Pythagorean theorem, parallel lines never meeting) hold throughout the entire space. This flatness is directly related to the global applicability of Cartesian coordinate systems and their orthogonal axes.

2.2.1 Operations in Euclidean Space: Where They Work and Where They Fall Short

Operations in Euclidean space are straightforward and intuitive due to their linear nature.

- **Where they work in robotics:**
 - **Translational Motion:** Representing and manipulating a robot's position in a fixed Cartesian coordinate system using 3D vectors. For instance, if a robot moves by a vector $\Delta \mathbf{p}$, its new position $\mathbf{p}' = \mathbf{p} + \Delta \mathbf{p}$ is a simple vector addition.

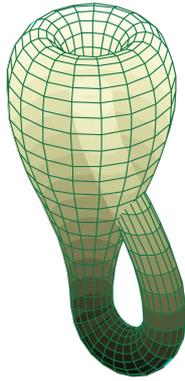
- **Robot Joint Angles:** For many types of robot joints (e.g., revolute or prismatic), their angles or displacements can be directly added or subtracted as scalar or vector quantities.
 - **Velocities and Accelerations:** Linear velocities and accelerations can be represented as vectors in \mathbb{R}^3 , and their operations (e.g., $\mathbf{v}' = \mathbf{v} + \mathbf{a}\Delta t$) follow Euclidean vector algebra.
- **Where they fall short for robotics:**
 - **Orientations (Rotations):** Attempting to add two rotation matrices (which are 3×3 real matrices) element-wise generally produces a matrix that is no longer a valid rotation matrix (i.e., it doesn't preserve length or handedness, and its determinant is not 1). For example, averaging multiple orientations by directly averaging their matrix elements will not yield a meaningful or valid average orientation.
 - **Poses (Rigid Body Transformations):** Similar to rotations, rigid body transformations (combinations of rotation and translation) cannot be directly added or linearly interpolated using standard Euclidean operations without violating their inherent geometric properties.
 - **Interpolation along curved paths:** Linearly interpolating between two orientations or poses in Euclidean space (e.g., averaging matrix components) does not guarantee a valid intermediate orientation/pose and often produces physically incorrect or non-smooth paths.

2.3 Manifold

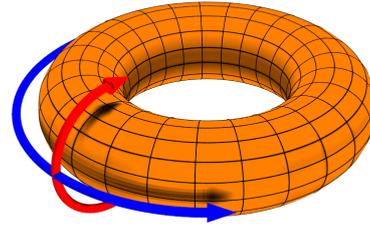
A *manifold* is a topological space that locally resembles Euclidean space. More precisely, for every point within a manifold, there exists a neighborhood around that point that is homeomorphic (topologically equivalent) to an open subset of \mathbb{R}^n . The integer n is known as the dimension of the manifold. While locally flat, manifolds can be globally curved, presenting a stark contrast to Euclidean spaces which are globally flat.

- **The Surface of a Sphere:** This is a 2D manifold embedded in 3D Euclidean space. Locally, any small patch on the surface appears flat, like a piece of a 2D plane. However, globally, the sphere is curved. Operations like walking in a "straight line" on the surface will eventually bring you back to your starting point, which is not characteristic of a flat plane.
- **The Perimeter of a Circle:** This is a 1D manifold embedded in 2D Euclidean space. Locally, any small segment of the circle appears as a straight line. Globally, it is a closed curve.
- **A Crumpled Piece of Paper:** If a piece of paper is crumpled without tearing or folding along sharp creases, its surface remains a 2D manifold. Despite its distorted appearance in 3D space, any small region on the paper can still be flattened out and locally mapped to a 2D plane.
- **Euclidean Space Itself (\mathbb{R}^n):** \mathbb{R}^n is also a manifold. In this case, every point's neighborhood is globally equivalent to \mathbb{R}^n , making it a manifold that is also flat everywhere.

The Earth provides a tangible example: while it is a sphere (a 2D manifold), we experience its surface as a flat Euclidean plane locally. For short distances, standard Euclidean geometry applies (e.g., measuring distances with a tape measure, defining local North-East-Down axes).



(a) Klein Bottle



(b) Torus

Figure 1: Interesting Manifolds. Mathematicians go crazy about these.

However, for long distances (e.g., intercontinental travel), the curvature of the Earth becomes apparent, and Euclidean plane geometry is insufficient.

When attempting to perform operations on elements of a curved manifold using traditional vector space operations (like vector addition), the result often lies "outside" the manifold. For instance, if you add two vectors originating from points on a sphere, their sum might point into or out of the sphere, rather than remaining on its surface. This is because standard vector addition presumes a flat ambient space, which does not respect the intrinsic curvature of the manifold. However, because manifolds are locally Euclidean, operations on an "infinitesimal patch" effectively behave as if they are in Euclidean space, enabling local approximations.

2.3.1 What is a Differentiable (Smooth) Manifold?

A differentiable (or smooth) manifold is a topological space that locally resembles Euclidean space, and where these local resemblances (called coordinate charts) are compatible in a smooth way. This means that if one transitions between different local coordinate systems on overlapping regions of the manifold, the transition functions are differentiable. This inherent smoothness allows for the consistent application of calculus (differentiation and integration) across the manifold.

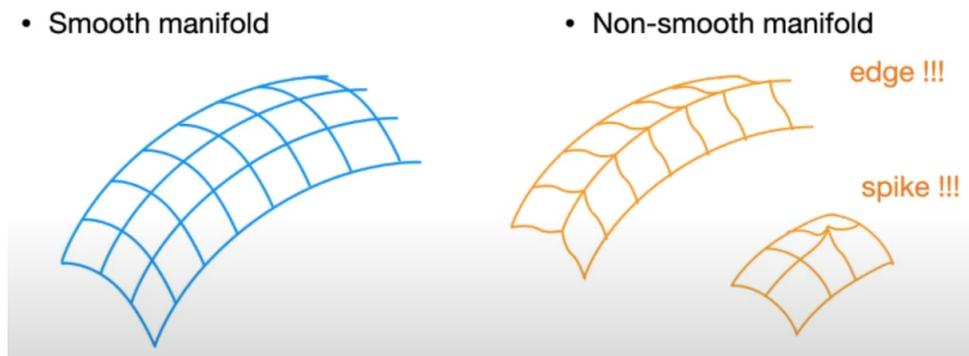


Figure 2: Non-smooth manifold has edges or spikes.

2.4 Field in Mathematics

In mathematics, a *field* is an algebraic structure where addition, subtraction, multiplication, and division are well-defined and behave much like they do for rational, real, or complex numbers.

Formally, a field F is a set equipped with two binary operations (addition and multiplication) that satisfy a set of axioms including commutativity, associativity, and distributivity, and importantly, the existence of additive and multiplicative identity elements, and additive and multiplicative inverses for all non-zero elements. Common examples of fields include the set of real numbers (\mathbb{R}) and the set of complex numbers (\mathbb{C}). In the context of vector spaces in robotics, the field of real numbers (\mathbb{R}) is almost universally used for scalar multiplication.

2.5 Rotation Representations

Representing the orientation of a rigid body in 3D space is a critical task in robotics. Various mathematical forms exist, each with its own advantages and disadvantages in terms of parametrization, interpretability, and computational properties. All these representations aim to capture the 3 **Degrees of Freedom (DOF)** inherent in 3D rotation. Rotations in 3D lie on a 3D manifold, which cannot be drawn on paper.

- **Rotation Matrix (Direction Cosine Matrix - DCM):** An orientation can be represented by a 3×3 matrix C (or R). This matrix has 9 elements. For C to represent a pure rotation, its columns (and rows) must form an orthonormal basis, and its determinant must be $+1$. These conditions impose constraints on the 9 elements, reducing the independent variables to 3 DOF. Operations like composing two rotations are straightforward (matrix multiplication).
- **Euler Angles:** Euler angles represent a 3D orientation using a sequence of three elemental rotations about fixed or rotating axes (e.g., X-Y-Z, Z-Y-X). This representation uses 3 parameters, directly corresponding to the 3 DOF of rotation. They are intuitive for human interpretation and control.
- **Quaternions:** A quaternion uses 4 parameters (q_w, q_x, q_y, q_z) to represent an orientation. To represent a rotation, the quaternion must be a "unit quaternion," meaning its norm (length) is 1 ($q_w^2 + q_x^2 + q_y^2 + q_z^2 = 1$). This unit norm condition imposes one constraint, effectively leaving 3 DOF ($4 - 1 = 3$). Quaternions are well-suited for interpolation and composition of rotations.
- **Rodrigues Vector (Axis-Angle Representation):** An orientation can be represented by a 3D vector where the direction of the vector defines the axis of rotation, and the magnitude of the vector defines the angle of rotation about that axis. This uses 3 parameters, corresponding to the 3 DOF.

2.5.1 Gimbal Lock

A significant challenge for minimal parameterizations of rotation (i.e., those using exactly 3 parameters, such as Euler angles and rotation vectors at certain angles) is *gimbal lock*. This phenomenon occurs when two of the rotation axes become aligned, leading to a loss of a degree of freedom and an infinite number of angle combinations representing the same physical orientation. For example, in a Z-Y-X Euler angle sequence, if the second rotation (Y-axis rotation, pitch) is $\pm 90^\circ$, the first and third axes (Z and X) become aligned. Any subsequent rotation about the Z-axis would be indistinguishable from a rotation about the X-axis, making it impossible to uniquely determine the full orientation from the angles alone. This mathematical singularity can cause computational issues in algorithms attempting to uniquely identify orientation parameters. Non-minimal representations, like unit quaternions and rotation matrices, do not suffer from gimbal lock.

2.6 Invariance and Equivariance

In mathematics and engineering, *invariance* and *equivariance* describe how properties or functions behave under transformations. These concepts are crucial for understanding the consistency and generality of solutions in various systems.

- **Invariance:** A property or function is *invariant* under a transformation if it remains unchanged after the transformation is applied. If a function f takes an input x and produces an output $f(x)$, and g is a transformation, then f is invariant under g if $f(g(x)) = f(x)$ for all valid x .
 - **Example:** The distance between two points in Euclidean space is invariant under rigid body transformations (translations and rotations). If you translate or rotate a pair of points, the distance between them remains the same. Similarly, the area of a polygon is invariant under rigid transformations.
- **Equivariance:** A function or property is *equivariant* under a transformation if applying the transformation to the input results in a predictable transformation of the output. This means the structure of the function is preserved, but its output values might change in a consistent manner relative to the transformation. If a function F takes an input x and produces $F(x)$, and g is a transformation, then F is equivariant under g if there exists a corresponding transformation g' such that $F(g(x)) = g'(F(x))$ for all valid x . Often, g' is the same type of transformation as g , or a transformation directly related to g .
 - **Example:** The centroid (geometric center) of a collection of points is equivariant under rigid body transformations. If you translate or rotate a set of points, the centroid of the new set will be the translation or rotation of the original centroid, respectively. That is, if \mathbf{P} is a set of points and $\text{centroid}(\mathbf{P})$ is its centroid, then $\text{centroid}(g(\mathbf{P})) = g(\text{centroid}(\mathbf{P}))$ for a rigid transformation g .

3 Lie Theory

The study of rigid body motion in robotics necessitates mathematical structures beyond traditional vector spaces. While Euclidean spaces are excellent for describing translational displacements, they fall short when representing and manipulating orientations and poses. This is where the theory of Lie groups becomes indispensable.

3.1 The Concept of a Group

Before defining Lie groups, it is essential to understand the fundamental concept of a *group* in abstract algebra. A group is a set equipped with a single binary operation that combines any two elements to form a third element within the same set, satisfying four specific axioms.

A *group* (G, \cdot) is a set G together with a binary operation $\cdot : G \times G \rightarrow G$ such that for all $a, b, c \in G$:

- (a) **Closure:** For any $a, b \in G$, the result of the operation $a \cdot b$ is also in G .
- (b) **Associativity:** The operation is associative, i.e., $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- (c) **Identity Element:** There exists a unique element $e \in G$, called the identity element, such that for every $a \in G$, $a \cdot e = e \cdot a = a$.
- (d) **Inverse Element:** For every $a \in G$, there exists a unique element $a^{-1} \in G$, called the inverse of a , such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

3.1.1 Examples of Groups

- **Integers under Addition:** $(\mathbb{Z}, +)$ is a group. The sum of any two integers is an integer (closure). Addition is associative. The identity element is 0. The inverse of any integer n is $-n$.
- **Non-zero Real Numbers under Multiplication:** $(\mathbb{R} \setminus \{0\}, \times)$ is a group. The product of any two non-zero real numbers is a non-zero real number. Multiplication is associative. The identity element is 1. The inverse of any non-zero real number x is $1/x$.
- **Symmetric Group (Permutations):** The set of all permutations of n distinct objects, with function composition as the operation, forms a group S_n . For example, S_3 consists of 6 permutations of 3 objects.

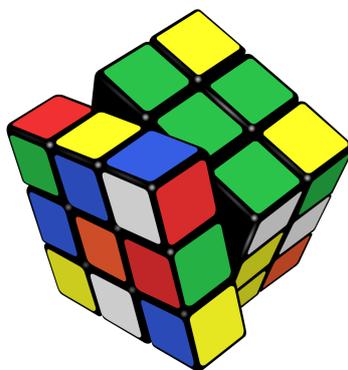


Figure 3: The manipulations of the Rubik's Cube form the Rubik's Cube group.

3.1.2 Matrix Groups (General Definition)

A **matrix group** is a group whose elements are matrices and whose group operation is matrix multiplication. For a set of matrices to form a group, they must satisfy the four group axioms (closure, associativity, identity element, inverse element) under matrix multiplication. This implies that all matrices in a matrix group must be invertible.

3.1.3 Examples of Matrix Groups

- **General Linear Group $\text{GL}_n(\mathbb{R})$** : This is the set of all $n \times n$ invertible matrices with real entries. A matrix $A \in \text{GL}_n(\mathbb{R})$ if and only if $\det(A) \neq 0$.
- **Orthogonal Group ($O(n)$)**: This group consists of all $n \times n$ real matrices C that are *orthogonal*. Orthogonal matrices satisfy the condition $CC^T = I$, where I is the identity matrix. This property ensures that the matrix preserves vector lengths and angles. For any orthogonal matrix, its determinant squared is 1, meaning $\det(C) = \pm 1$. The orthogonal group thus includes both transformations that represent proper rotations and transformations that represent reflections. $O(n) = O(n)^+ \cup O(n)^-$
- **Special Orthogonal Group ($\text{SO}(n) = O(n)^+$)**: This is a subgroup of the Orthogonal Group. It comprises all $n \times n$ orthogonal matrices C that have a *determinant of +1* ($\det(C) = 1$). The term "special" refers specifically to this determinant constraint. This condition ensures that the transformation is a "proper rotation," preserving both vector lengths and the orientation (or handedness) of the coordinate system, thereby excluding reflections.
- **Euclidean Group ($E(n)$)**: This group represents all rigid transformations (isometries) of n -dimensional Euclidean space that preserve the Euclidean distance between points. These transformations can include rotations, reflections, translations, and any combination thereof. It is the group of all transformations that map a grid onto itself while preserving distances between grid points.
- **Special Euclidean Group ($\text{SE}(n)$)**: This is a subgroup of the Euclidean Group. It consists of all rigid transformations that preserve Euclidean distance *and* also preserve the **orientation** (or handedness) of the space. This means it includes only proper rotations (those with a determinant of +1) and translations, along with their combinations. It explicitly excludes reflections. For robotics, $\text{SE}(3)$ is of paramount interest, as a robot's motion involves continuous, rigid transformations (poses) without reflections. Elements of $\text{SE}(3)$ are 4×4 matrices where the upper-left 3×3 block is a rotation matrix from $\text{SO}(3)$ and the upper-right block is a translation vector.
- **Affine Group $\text{Aff}_n(\mathbb{R})$** : This group comprises $(n + 1) \times (n + 1)$ matrices of the form $\begin{pmatrix} A & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix}$, where $A \in \text{GL}_n(\mathbb{R})$ is an $n \times n$ invertible matrix and $\mathbf{t} \in \mathbb{R}^n$ is an $n \times 1$ translation vector. An element of $\text{Aff}_n(\mathbb{R})$ acts on a vector $\mathbf{x} \in \mathbb{R}^n$ (represented homogeneously as $\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}$) to produce an affine transformation $\begin{pmatrix} A\mathbf{x} + \mathbf{t} \\ 1 \end{pmatrix}$.

3.2 Definition of a Lie Group

A *Lie group* is a mathematical object that elegantly combines the abstract algebraic structure of a group with the geometric properties of a smooth manifold. Specifically, a Lie group G is a smooth manifold whose elements satisfy the group axioms. In other words, Lie group is a group that is also a differentiable (or smooth) manifold, such that its group operations (multiplication and inversion) are smooth maps. This "smoothness" means that these operations are infinitely differentiable, which is crucial for applying calculus to group elements.

Lie Group = Smooth Manifold + Matrix Group

3.2.1 Special Orthogonal Group $\text{SO}(3)$

The Special Orthogonal Group $\text{SO}(3)$ is the Lie group representing all possible 3D rotations about a fixed point (the origin). Its elements are 3×3 real matrices C (also known as Direction Cosine Matrices or DCMs) that satisfy two conditions:

- **Orthogonality:** $CC^T = I$, where I is the 3×3 identity matrix. This implies that the columns (and rows) of C are orthonormal vectors.
- **Positive Determinant:** $\det(C) = 1$. This condition distinguishes proper rotations from improper rotations (which include reflections and have a determinant of -1).

$\text{SO}(3)$ forms a group under matrix multiplication:

- **Closure:** If $C_1, C_2 \in \text{SO}(3)$, then $C_1C_2 \in \text{SO}(3)$. This holds because $(C_1C_2)(C_1C_2)^T = C_1C_2C_2^TC_1^T = C_1IC_1^T = C_1C_1^T = I$, and $\det(C_1C_2) = \det(C_1)\det(C_2) = 1 \cdot 1 = 1$.
- **Associativity:** Matrix multiplication is inherently associative, $(C_1C_2)C_3 = C_1(C_2C_3)$.
- **Identity:** The identity matrix $I \in \text{SO}(3)$ ($II^T = I$, $\det(I) = 1$) serves as the identity element.
- **Inverse:** For any $C \in \text{SO}(3)$, its inverse is $C^{-1} = C^T$. Since C^T is also orthogonal ($C^T(C^T)^T = C^TC = I$) and $\det(C^T) = \det(C) = 1$, C^{-1} is also in $\text{SO}(3)$.

3.2.2 Special Euclidean Group $\text{SE}(3)$

The Special Euclidean Group $\text{SE}(3)$ is the Lie group representing all possible 3D rigid body transformations (or poses), which combine a rotation and a translation. Its elements are 4×4 matrices T of the form:

$$T = \begin{bmatrix} C & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

where $C \in \text{SO}(3)$ is the 3×3 rotation matrix and $\mathbf{r} \in \mathbb{R}^3$ is the 3×1 translation vector. $\mathbf{0}^T = [0, 0, 0]$ is a 1×3 row vector of zeros. $\text{SE}(3)$ forms a group under matrix multiplication:

- **Closure:** If $T_1, T_2 \in \text{SE}(3)$, then $T_1T_2 \in \text{SE}(3)$. This is shown by:

$$T_1T_2 = \begin{bmatrix} C_1 & \mathbf{r}_1 \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} C_2 & \mathbf{r}_2 \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} C_1C_2 & C_1\mathbf{r}_2 + \mathbf{r}_1 \\ \mathbf{0}^T & 1 \end{bmatrix}$$

Since $C_1C_2 \in \text{SO}(3)$ and $C_1\mathbf{r}_2 + \mathbf{r}_1 \in \mathbb{R}^3$, the product is also in $\text{SE}(3)$.

- **Associativity:** Matrix multiplication is associative.
- **Identity:** The 4×4 identity matrix $I_4 = \begin{bmatrix} I_3 & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \in \text{SE}(3)$ is the identity element.
- **Inverse:** For any $T \in \text{SE}(3)$, its inverse is $T^{-1} = \begin{bmatrix} C^T & -C^T\mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix}$. Since $C^T \in \text{SO}(3)$ and $-C^T\mathbf{r} \in \mathbb{R}^3$, T^{-1} is also in $\text{SE}(3)$.

3.2.3 Differentiable Manifold Property

To distinguish Lie groups from regular (non-Lie) groups, it is crucial to demonstrate their *differentiable manifold* property. This involves showing that they are smooth manifolds and that their group operations (multiplication and inversion) are smooth (differentiably continuous) maps.

For $\text{SO}(n)$ (e.g., $\text{SO}(3)$ for 3D rotations):

- **As a Smooth Manifold:** $\text{SO}(n)$ is a subset of $\mathbb{R}^{n \times n}$ (the space of all $n \times n$ real matrices). The defining conditions for $\text{SO}(n)$ ($CC^T = I$ and $\det(C) = 1$) are smooth (differentiable) equations. For instance, the orthogonality condition $CC^T = I$ imposes $n(n+1)/2$ independent quadratic equations on the n^2 entries of C . These equations define a smooth surface within the higher-dimensional space $\mathbb{R}^{n \times n}$. Locally, around any rotation matrix $C_0 \in \text{SO}(n)$, the space of small perturbations can be smoothly mapped to a Euclidean space (its Lie algebra $\mathfrak{so}(n)$), which serves as the tangent space at C_0 . This local Euclidean behavior, combined with the smoothness of its defining constraints, confirms $\text{SO}(n)$ as a smooth manifold.
- **Smooth Group Operations:**
 - **Multiplication:** Matrix multiplication is a polynomial function of the matrix entries. Polynomials are infinitely differentiable (smooth). Thus, if $C_1, C_2 \in \text{SO}(n)$, their product $C_1 C_2$ is computed via smooth operations, making the group multiplication a smooth map.
 - **Inversion:** For any $C \in \text{SO}(n)$, its inverse is $C^{-1} = C^T$. The transpose operation is a linear (and therefore smooth) transformation of the matrix entries.

Since both group multiplication and inversion are smooth operations, $\text{SO}(n)$ fully satisfies the definition of a Lie group.

For $\text{SE}(n)$ (e.g., $\text{SE}(3)$ for 3D poses):

- **As a Smooth Manifold:** $\text{SE}(n)$ is a subset of $\mathbb{R}^{(n+1) \times (n+1)}$. Its elements $T = \begin{smallmatrix} C & \mathbf{r} \\ \mathbf{0}^T & 1 \end{smallmatrix}$ are constrained by $C \in \text{SO}(n)$ and the fixed bottom row $[0, \dots, 0, 1]$. The constraints on C are smooth, and the linear constraints on the bottom row are also smooth. These smooth conditions collectively define $\text{SE}(n)$ as a smooth manifold. Locally, the space of small perturbations around any $T_0 \in \text{SE}(n)$ can be smoothly mapped to its Lie algebra $\mathfrak{se}(n)$, which is a Euclidean space.
- **Smooth Group Operations:**
 - **Multiplication:** The product of two matrices in $\text{SE}(n)$ is matrix multiplication, which is a smooth operation. For $T_1 = \begin{smallmatrix} C_1 & \mathbf{r}_1 \\ \mathbf{0}^T & 1 \end{smallmatrix}$ and $T_2 = \begin{smallmatrix} C_2 & \mathbf{r}_2 \\ \mathbf{0}^T & 1 \end{smallmatrix}$, their product is $T_1 T_2 = \begin{smallmatrix} C_1 C_2 & C_1 \mathbf{r}_2 + \mathbf{r}_1 \\ \mathbf{0}^T & 1 \end{smallmatrix}$. The resulting $C_1 C_2$ and $C_1 \mathbf{r}_2 + \mathbf{r}_1$ are smooth functions of the original matrix entries.
 - **Inversion:** The inverse of $T \in \text{SE}(n)$ is $T^{-1} = \begin{smallmatrix} C^T & -C^T \mathbf{r} \\ \mathbf{0}^T & 1 \end{smallmatrix}$. Both C^T and $-C^T \mathbf{r}$ are smooth functions of C and \mathbf{r} .

The smoothness of these group operations confirms that $\text{SE}(n)$ is a Lie group.

3.2.4 Distinction from Regular Groups

The critical distinction between Lie groups and general (algebraic) groups lies in the presence of this **smooth manifold structure** and the **smoothness of their group operations**.

- A regular group like $(\mathbb{Z}, +)$ (integers under addition) is a group, but it is a discrete set of points, not a continuous manifold. One cannot define derivatives or tangent spaces for it in the same way as for Lie groups.

- Even some matrix groups may not be Lie groups if their elements are defined over a discrete field (e.g., rational numbers), which would prevent the definition of a smooth manifold and differentiable operations.

The differentiable manifold property of $SO(n)$ and $SE(n)$ means they are not merely abstract sets with operations, but continuous, geometrically well-behaved spaces where the tools of calculus can be consistently applied. This characteristic is fundamental for modeling continuous motion and transformations in robotics.

3.2.5 Group Action: Definition

- A group can **act** on another set \mathcal{V} to **transform** its elements.
- Given $\mathbf{X}, \mathbf{Y} \in \mathcal{G}$ and $\mathbf{v} \in \mathcal{V}$, the action ‘ \cdot ’ is such that:
 - **Identity** is the null action: $\mathbf{E} \cdot \mathbf{v} = \mathbf{v}$
 - It is **compatible** with composition: $(\mathbf{X} \cdot \mathbf{Y}) \cdot \mathbf{v} = \mathbf{X} \cdot (\mathbf{Y} \cdot \mathbf{v})$
- Lie groups were formerly known as "*Continuous Transformation groups*".

4 The Bridge to Linearity: Lie Algebra

The *Lie algebra* is intimately associated with its corresponding Lie group and serves as a fundamental bridge to enable linear operations within the context of nonlinear Lie groups. It captures the infinitesimal structure of the Lie group.

$$\text{Lie Group} \iff \text{Lie Algebra} \iff \text{Cartesian Coordinate}$$

4.0.1 Why Cannot We Directly Work on Lie Groups?

While Lie groups like $SO(3)$ and $SE(3)$ provide elegant, singularity-free representations for orientations and poses, their non-Euclidean geometry means we cannot directly apply standard vector space operations (such as vector addition or component-wise averaging) to their elements.

- **Not Vector Spaces:** Lie groups do not satisfy all the axioms of a vector space. For example, they are not closed under vector addition: the sum of two rotation matrices $C_1 + C_2$ generally yields a matrix that is neither orthogonal nor has a determinant of 1, and thus is not a valid rotation matrix. Similarly, the zero matrix, which is the additive identity in a vector space, is not an element of $SO(3)$ or $SE(3)$. Imagine the surface of the Earth as our Lie Group, G . The "elements" of this group are the cities (or any point on the surface). We can't simply add the coordinates of New York and Tokyo to get a valid location on Earth; the space is curved.
- **Geometric Distortion:** Applying Euclidean operations directly on Lie group elements distorts their inherent geometric meaning. For instance, linearly interpolating between two rotation matrices by averaging their corresponding matrix elements does not produce a valid rotation at intermediate steps and results in non-physical paths.
- **Calculus Limitations:** Standard calculus tools (e.g., gradients for optimization, linear approximations via Taylor series) are built on the linearity of Euclidean spaces. All the calculus we have in Euclidean Space. Directly differentiating functions on curved manifolds is complex and requires specialized machinery. (Differential Geometry Course for further theory.)

This challenge leads to the necessity of Lie algebras, which provide a linearized "tangent space" where Euclidean operations can be applied locally.

4.0.2 Formal Definition

Formally, a *Lie algebra* \mathfrak{g} (corresponding to a Lie group G) is a vector space over a field F (typically \mathbb{R}) equipped with a binary operation called the *Lie bracket*, denoted $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$. The Lie bracket must satisfy properties of bilinearity, alternating property ($[X, X] = 0 \implies [X, Y] = -[Y, X]$), and the Jacobi identity.

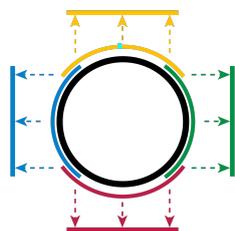
4.0.3 Tangent Space and Linearization

The Lie algebra can be intuitively understood as the *tangent space* of the Lie group at its identity element.

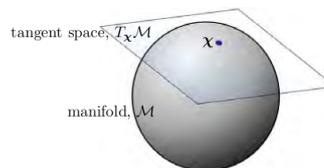
Intuitive Analogy: Imagine the surface of the Earth as our **Lie Group** (G), a curved manifold. The "elements" of this group are the cities. Now, pick a reference point the **identity element** (e). Let's choose Greenwich, UK.

The **tangent space** is a flat plane that just touches the globe at Greenwich. This plane represents all possible directions and speeds you could travel *from* Greenwich. A vector on this plane is a velocity. This flat plane of velocities *is* the **Lie algebra** (\mathfrak{g}).

Critically, the structure of this flat tangent space at the identity contains all the information needed to understand the local geometry of the *entire* curved group. We can use the group's properties to "move" this flat map to any other city on the globe.



(a) Tangent Space on Circle (1D Manifold)



(b) Tangent Space on Sphere (2D Manifold)

Figure 4: Tangent Space of Smooth Manifolds

- **What "Tangent Space" Means:** Imagine the curved surface of a manifold. At any specific point on this surface, you can place a flat plane that "just touches" the surface at that single point. This flat plane is the the tangent space at that point. It's a vector space, meaning you can perform linear operations (addition, scalar multiplication) within it, which **allows us to apply linear algebra and calculus, tools that are not directly applicable on the curved manifold of the Lie group.** They do not explain the entire space but some part of it. Consider Euler Angles, they explain until the gimbal lock.
- **Why the Identity Element is Special:** It is convenient by making the calculation easier. Also, every group has an identity element by definition.
- **Analogy to Linearization (Taylor Approximation):** Just as a tangent line (1D tangent space) is the best linear approximation of a curve at a point, the Lie algebra provides the best linear approximation of the Lie group manifold around its identity element. This is analogous to how Taylor series approximations linearize a nonlinear function by approximating it with a polynomial (often just the linear term) around a specific point. Small movements within the Lie algebra (a vector space) accurately reflect infinitesimal movements on the Lie group manifold. **We generally treat vectors on Lie Algebra as the small perturbations $\delta\phi$ (SO(3)) or $\delta\xi$ SE(3), similar to linearization error.**

4.1 Derivation of the Tangent Space and Skew-Symmetric Matrices

Our motivation for understanding the tangent space stems from the need to analyze the instantaneous motion of an element (like a rotation matrix) on a Lie group. The velocity vector of a point moving on such a manifold does not reside directly within the manifold itself but rather in its associated tangent space at that point.

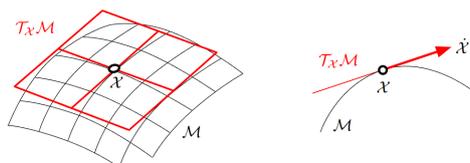


Figure 5: Velocity vector is key to derive tangent space.

We are not directly interested in the velocity vector. Rather, we use it as a portal to move to the tangent space by physical intuition.

Any vector in the cartesian coordinates can be converted to 3×3 skew-symmetric matrix can be uniquely represented in the form:

$$[\omega]_{\times} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

where $\omega = (\omega_1, \omega_2, \omega_3)^T \in \mathbb{R}^3$ is a 3D vector. This operation is known as the skew-symmetric operator or "hat" map $(\cdot)^\wedge$. This skew-symmetric matrix is the Lie Algebra correspondence of vector ω which is in the vector space where calculus is available. (See appendix for proof)

This matrix form signifies that the set of all possible instantaneous velocities at the identity forms the Lie algebra of $SO(3)$, denoted as $\mathfrak{so}(3)$, and its elements are precisely these 3×3 skew-symmetric matrices. Since $[\omega]_{\times}$ has 3 independent components, the dimension of $SO(3)$ (and its Lie algebra) is 3, corresponding to the three degrees of freedom of rotation in 3D space.

This derivation shows how every velocity vector in the tangent space of $SO(3)$ can be represented by a skew-symmetric matrix, providing a linear space where calculus can be effectively applied.

4.2 Cartesian Coordinates to Lie Algebra

The Lie algebra $\mathfrak{so}(3)$ consists of 3×3 skew-symmetric matrices. A generic element $\omega_x \in \mathfrak{so}(3)$ is represented as:

$$\omega_x = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \in \mathfrak{so}(3)$$

This skew-symmetric matrix can be uniquely decomposed into a linear combination of basis generators for $\mathfrak{so}(3)$:

$$\omega_x = \omega_x \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \omega_y \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \omega_z \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Each of these basis matrices corresponds to a rotation around the x, y, and z axes, respectively.

Conversely, any vector $\omega = [\omega_x, \omega_y, \omega_z]^T \in \mathbb{R}^3$ in Cartesian space can be seen as the set of coefficients for this linear combination. This establishes a direct isomorphism between the skew-symmetric matrices in $\mathfrak{so}(3)$ and vectors in \mathbb{R}^3 . This mapping is often denoted by the "hat" $(\cdot)^\wedge$ or "vee" $(\cdot)^\vee$ operators, allowing us to seamlessly convert between the abstract Lie algebra elements and the more familiar Cartesian vector representation for computational purposes.

4.3 The Exponential and Logarithmic Map

The *exponential map* provides the fundamental connection from the Lie algebra to the Lie group. It allows us to "exponentiate" a Lie algebra element (a small linear change) to obtain a corresponding Lie group element (a rotation or transformation).

$$\text{Lie Group} \xleftarrow{\text{exp}} \text{Lie Algebra} \xleftarrow{\text{hat}} \text{Cartesian}$$

Since we generally do all the calculus in the cartesian coordinates (Euclidean space), most of the time we only use exponential map where we do calculation in the cartesian coordinates then transfer it to the tangent space and then use exponential map to map it onto lie group.

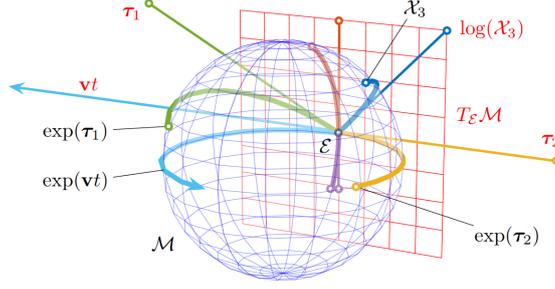


Figure 6: Lie Group (2D Manifold / Sphere Surface), Lie algebra (Tangent Space) and mapping between them (exp, log). Vectors on tangent plane are wrapped over the sphere surface (manifold) by exponential map.

4.3.1 Exponential Map

exp map arises naturally by considering the time-derivatives of $\mathcal{X} \in \mathcal{M}$ over the manifold:

$$\dot{\mathcal{X}} = \mathcal{X} \mathbf{v}^\wedge$$

This ODE (comes from the derivation of skew symmetric, check Eqn. 9 in [4]) has a solution in from

$$\mathcal{X}_t = \mathcal{X}_0 e^{\mathbf{v}^\wedge t} = \mathcal{X}_0 \exp(\mathbf{v}^\wedge t)$$

Similarly

$$\exp(\mathbf{v}^\wedge t) = \mathcal{X}_0^{-1} \mathcal{X}_t$$

, which satisfies the group property, so $\exp(\mathbf{v}^\wedge t)$ must be a group element as well.

The exponential map from a matrix Lie algebra \mathfrak{g} to its Lie group G is defined as the *matrix exponential* (a type of Taylor Series):

$$\exp(A) = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}A^n$$

where A is an element of the Lie algebra. $\exp(A)$ converges (See Appendix for proof).

- For $SO(3)$: $C = \exp(\phi^\wedge)$ for $C \in SO(3)$ and $\phi^\wedge \in \mathfrak{so}(3)$, where ϕ is the angle in vector space representation. This mapping is surjective-only, meaning multiple Lie algebra elements can map to the same Lie group element.
- For $SE(3)$: $T = \exp(\xi^\wedge)$ for $T \in SE(3)$ and $\xi^\wedge \in \mathfrak{se}(3)$, where ξ is the pose.

4.3.2 Logarithmic Map

The matrix logarithm is the inverse operation, mapping elements from the Lie group back to the Lie algebra.

- For $SO(3)$: $\phi = (\log(C))^\vee$.
- For $SE(3)$: $\xi = (\log(T))^\vee$.

5 Calculus in Lie Theory

As stated above, all the calculus we have in Euclidean space. So we do all the calculus in the vector space and then convert it to the lie algebra equivalent and then wrap onto the lie group.

5.1 Some Definitions (Joan Sola's Notation)

- **Plus Operator (\oplus):** The "plus" operator defines how a Lie group element \mathcal{X} is composed with a perturbation τ (an element from the Lie algebra's corresponding Cartesian space), resulting in a new Lie group element. This typically represents applying a small transformation to an existing pose or rotation. The right "plus" operator (perturbation on the right) is defined as:

$$\mathcal{X} \oplus \tau \triangleq \mathcal{X} \circ \text{Exp}(\tau) = \mathcal{X} \circ \exp(\tau^\wedge)$$

The symbol \circ denotes the group's composition operation (e.g., matrix multiplication).

- **Minus Operator (\ominus):** The "minus" operator quantifies the "difference" between two Lie group elements \mathcal{Y} and \mathcal{X} , expressing this difference as an element in the Lie algebra's corresponding Cartesian space. This is essential for calculating errors or relative transformations. For the right "minus" operator, where the difference is expressed in \mathcal{X} 's local frame, it is defined as:

$$\mathcal{Y} \ominus \mathcal{X} \triangleq \text{Log}(\mathcal{X}^{-1} \circ \mathcal{Y})$$

This operator returns a Lie algebra element (a vector in its corresponding Cartesian space).

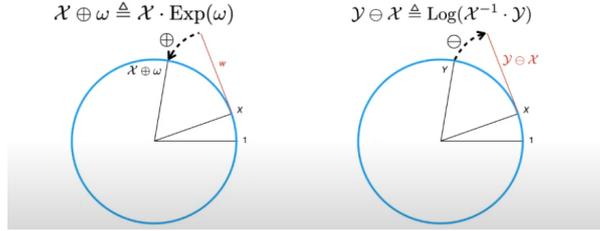


Figure 7: Plus and minus on $\text{SO}(3)$

- **Jacobian on Euclidean Space:** This is the standard definition of a Jacobian matrix for a function $f(\mathbf{x})$ mapping from one Euclidean vector space to another, where \mathbf{h} represents a vector perturbation.

$$\mathbf{J} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \triangleq \lim_{\mathbf{h} \rightarrow 0} \frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})}{\mathbf{h}} \in \mathbb{R}^{n \times m}$$

- **Jacobian on Lie Groups:** This definition extends the concept of a Jacobian to functions $f(\mathcal{X})$ where the input \mathcal{X} and output $f(\mathcal{X})$ live on Lie groups (or manifolds). It leverages the "plus" (\oplus) and "minus" (\ominus) operators to define perturbations and differences in the respective tangent spaces, allowing for computation using vector calculus in those spaces.

$$\mathbf{J} = \frac{Df(\mathcal{X})}{D\mathcal{X}} \triangleq \lim_{\tau \rightarrow 0} \frac{f(\mathcal{X} \oplus \tau) \ominus f(\mathcal{X})}{\tau} \in \mathbb{R}^{n \times m}$$

5.2 Derivative of Rotation Group Action on a Vector

Here, we derive the derivative of a function that defines the action of the rotation group $\text{SO}(3)$ on a 3D vector. This is a fundamental operation in robotics, particularly for analyzing how uncertainties in orientation affect vector quantities.

The action of the rotation group $\text{SO}(3)$ on a vector $\mathbf{p} \in \mathbb{R}^3$ is defined by the function:

$$f : \text{SO}(3) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \quad ; \quad (\mathcal{R}, \mathbf{p}) \mapsto f(\mathcal{R}, \mathbf{p}) = \mathcal{R}\mathbf{p}$$

This function takes a rotation matrix \mathcal{R} and a 3D point \mathbf{p} , and returns the rotated 3D point.

The derivative of this function with respect to a perturbation on the rotation \mathcal{R} (expressed as an infinitesimal Lie algebra element $\boldsymbol{\theta}$) is calculated using the Lie group derivative definition. This derivative describes how a small change in rotation affects the transformed vector.

$$\begin{aligned} \frac{Df}{D\mathcal{R}} &= \lim_{\boldsymbol{\theta} \rightarrow 0} \frac{f(\mathcal{R} \oplus \boldsymbol{\theta}) - f(\mathcal{R})}{\boldsymbol{\theta}} && \text{Definition of Lie Group Jacobian} \\ &= \lim_{\boldsymbol{\theta} \rightarrow 0} \frac{(\mathcal{R} \circ \text{Exp}(\boldsymbol{\theta})) \cdot \mathbf{p} - \mathcal{R}\mathbf{p}}{\boldsymbol{\theta}} && \text{Substitute } f(\mathcal{R}, \mathbf{p}) = \mathcal{R}\mathbf{p} \text{ and } \mathcal{R} \oplus \boldsymbol{\theta} = \mathcal{R} \circ \text{Exp}(\boldsymbol{\theta}) \\ &\approx \lim_{\boldsymbol{\theta} \rightarrow 0} \frac{\mathcal{R} \cdot (\mathbf{I} + \boldsymbol{\theta}_\times) \cdot \mathbf{p} - \mathcal{R}\mathbf{p}}{\boldsymbol{\theta}} && \text{Small angle approximation: } \text{Exp}(\boldsymbol{\theta}) \approx \mathbf{I} + \boldsymbol{\theta}_\times \\ &= \lim_{\boldsymbol{\theta} \rightarrow 0} \frac{\mathcal{R} \cdot \boldsymbol{\theta}_\times \cdot \mathbf{p}}{\boldsymbol{\theta}} && \text{Simplify terms} \\ &= \lim_{\boldsymbol{\theta} \rightarrow 0} \frac{-\mathcal{R} \cdot \mathbf{p}_\times \cdot \boldsymbol{\theta}}{\boldsymbol{\theta}} && \text{Identity: } \boldsymbol{\theta}_\times \mathbf{p} = -\mathbf{p}_\times \boldsymbol{\theta} \\ &= -\mathcal{R} \cdot \mathbf{p}_\times && \text{Evaluate the limit} \end{aligned}$$

The result, $-\mathcal{R} \cdot \mathbf{p}_\times$, provides the Jacobian matrix that linearly relates infinitesimal changes in the rotation (expressed in the tangent space) to the corresponding changes in the rotated vector. This is essential for optimization and uncertainty propagation in rigid body mechanics.

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